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Site percolation problems and multi-site Potts models

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Abstract. The correspondence discovered by Kunz and Wu (1978) between the site percolation problem and the multi-site Potts problem is extended to several lattices. It is found that the site percolation problem can be very simply formulated in terms of the operators used by Temperley and Lieb (1971). In three instances applying the dual transformation to these operators induces the matching transformation found by Sykes and Essam (1964) and it is conjectured that this may hold generally. It follows that site percolation problems, like bond percolation problems, depend on finding eigenvalues of row transfer operators.

1. Introduction

Kunz and Wu (1978) have shown that site percolation problems on various planar lattices are closely related to multi-site Potts problems on related lattices. Each percolation site is replaced by a 'city' of surrounding sites. One possibility is to choose a percolation site at the centre of each bond of the Potts lattice. The Potts sites are coloured with q colours and we give the 'city' of Potts sites a weight $(1+g)$ if all its sites are coloured alike, otherwise a weight 1. We also introduce a 'magnetic field', that is to say we give a weight e^L to any site on the Potts lattice that is coloured with a particular colour (red say). Let $\psi(q, g, L)$ be the corresponding generating function for this Potts problem. A city with all Potts sites coloured alike is identified with a 'black' site in the percolation problem; a city with any other colouring is identified with a 'white' site in the percolation problem. If two 'black' sites are neighbours in the percolation lattice, the corresponding cities of Potts sites have exactly one Potts site in common, namely that corresponding to the bond. Using this fact Kunz and Wu defined a 'free energy' as

$$\left[\frac{\partial}{\partial q}\ln \psi(q, g, L)\right]_{q=1} \qquad (1)$$

and showed that the percolation probability and the mean-square size of cluster are related to the 'spontaneous magnetisation' and the 'zero-field susceptibility', that is to the first and second derivatives of the 'free energy' (1) with respect to L in the limit $L \to 0$. In their formulae, the probability p that a site is black is to be identified with the ratio $g/(1+g)$.

Baxter *et al* (1978) have shown that one particular multi-site Potts problem, a plane triangular lattice with three-site Potts-type interactions around alternate triangles, has an extremely simple formulation in terms of the operators introduced by Temperley

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and Lieb (1971). For $q = 2$ it reduces to the Ising problem on the plane triangular lattice, but we did not succeed in solving it for general q. We were able to locate the critical value of g (this is simply equal to q, the number of colours.) We show that this problem, in the limit $q \to 1$, is precisely the site percolation problem on the plane triangular lattice. It was found by Sykes and Essam (1964), that the critical value of p for this problem is $\frac{1}{2}$ and our work agrees with this.

We shall also show that the site percolation problems for various other lattices considered by Sykes and Essam (1964) can also be formulated, though not solved, in terms of eigenvalue problems involving simple looking Temperley–Lieb operators and that the relationship between a pair of matching lattices found by Sykes and Essam (1964) is equivalent to the 'dual' transformation applied to the Temperley–Lieb operators. Thus, the 'site' problems as well as the 'bond' problems can now be formulated as eigenvalue problems involving Temperley–Lieb operators. (As shown by Temperley and Lieb (1971), the bond percolation problem only needs operators involving two-site interactions between neighbouring sites.)

It is also known (Temperley and Lieb 1971, Baxter 1973) that the generating function for the bond percolation and the two-site Potts problems can be expressed as a Whitney–Tutte polynomial, that is to say as a sum over all the subgraphs obtained by removing bonds from the lattice but leaving all the sites. The generating function is

$$\varphi(q, f) = \sum q^c f^l \tag{2}$$

where $(1+f)$ is the weight given to a neighbouring pair of sites coloured alike (an unlike neighbouring pair being given a weight unity). In each subgraph, c is the number of separate components, an isolated site being reckoned as one component, and l is the number of bonds the subgraph contains. For the bond percolation problem the expected number of separate components is

$$\lim_{q \to 1} \frac{\partial}{\partial q}(\ln \varphi(q, f)). \tag{3}$$

This is because, for $q = 1$, $\varphi(q, f)$ reduces trivially to $(1+f)^L$, where L is the total number of bonds in the lattice, so (3) reduces to

$$\sum c f^l/(1+f)^L \tag{4}$$

that is, to the expected number of separate components when the probability that any given bond is active is $f/(1+f)$ and the probability that it is not active is $1/(1+f)$.

Kunz and Wu's result (1978) is that the site percolation problem is related to the corresponding limit of a Potts problem involving multi-site interactions. This problem can also be expressed as a 'hypergraph' expansion analogous to (2), in which we assign weights to the elementary triangles or squares of the lattice contained by each subgraph.

2. Subgraph expansion for the site percolation problem

Instead of (2) our generating function is

$$\psi(q, g) = \sum g^n q^c \tag{5}$$

where the sum is now over all the subgraphs obtained by 'blackening' one or more sites of the lattice. n is the number of black sites in the subgraph and we imagine all bonds on the lattice that connect neighbouring black points to be drawn in and c is the number of black connected components. (An isolated black site is counted as a component.) The expected number of components is obtained from (5) by differentiating $\ln \psi$ with respect to q, afterwards letting $q \to 1$. In this limit, ψ itself reduces to $(1+g)^N$ where N is the total number of sites in the lattice. The probability that a given site is blackened is then $g/(1+g)$.

We state the following result without formal proof. Consider the medial lattice L_M formed from our planar lattice L by taking the mid-points of each bond as sites in L_M and joining up such neighbouring mid-points.

In figure 1, let the dots correspond to blackened sites on L and the crosses to sites on L_M, both being plane square lattices. Then there is 1–1 correspondence between graphs formed by blackening sites on L and hypergraphs on L_M formed by combining one or more shaded squares. (Only alternate squares on L_M are shaded). Furthermore, the existence of a bond between two neighbouring blackened sites on L corresponds to a site common to two neighbouring shaded squares of L_M.

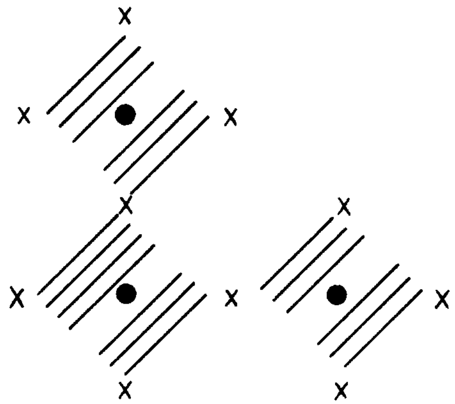

Figure 1

For the plane triangular and plane hexagonal lattices the medial lattice is the kagomé lattice.

Figure 2(a) illustrates the fact that each blackened site (dot) in the plane triangular lattice is surrounded by an elementary *hexagon* in the kagomé lattice L_M (crosses). Figure 2(b) illustrates the correspondence between blackened sites in the plane honeycomb lattice and elementary *triangles* of L_M.

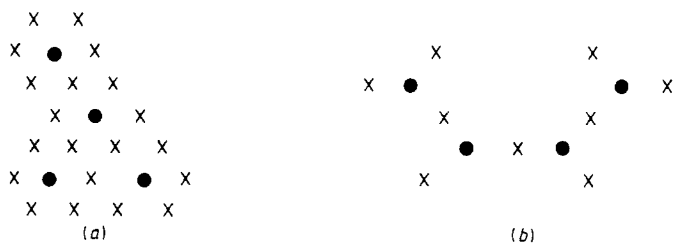

Figure 2

There is, however, another way in which we can associate a 'surrounding' lattice with possible ways of blackening sites in the plane triangular lattice. In the place of L_M we can use L' obtained by taking the central points of all the 'up' triangles in L' as sites of L.

In figure 3 dots represent a typical blackening of some of the sites of L and a corresponding choice of 'down' elementary triangles (shaded) in L', the vertices of which are marked by crosses.

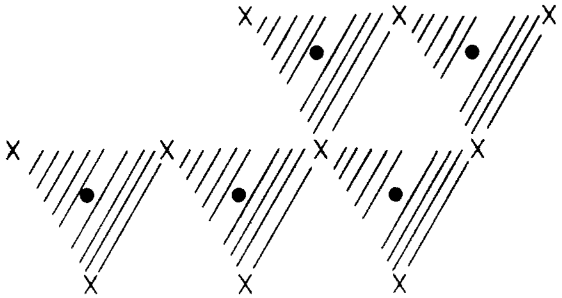

Figure 3

We can therefore say that if L is the plane square, honeycomb or triangular lattices we can find 'surrounding' lattices such that a point of L corresponds to an elementary polygon of the surrounding lattice. In other words, there is a correspondence between a choice of sites of L and modes of colouring L' or L_M such that all the sites in certain elementary polygons of L' or L_M are coloured alike. For elementary polygons that are triangles or squares it is, as we shall now show, relatively easy to write down operators that give a weight of $(1+g)$ to colourings such that all the sites in an elementary polygon are coloured alike and a weight of 1 for all other colourings of the sites in this polygon.

3. Row transfer operator form of the multi-site Potts problem

Various representations of the Temperley–Lieb operators are given by Baxter *et al* (1978, § 5). We shall need only the following properties.

$q^{1/2}U_{12}$: Attention is transferred from site A to site A'. Regardless of the colouring of site A, site A' in the next row is coloured with each of the q colours with equal probability. We may represent the effect thus:

$$R_A \to R_{A'} + Y_A + G_{A'} + \ldots, \qquad Y_A \to R_{A'} + Y_{A'} + G_{A'} + \ldots, \qquad \text{etc.}$$

$I_{AA'}$: Sites A and A' are coloured alike:

$$R_A \to R_{A'}, \qquad Y_A \to Y_{A'} \qquad \text{etc.}$$

$q^{1/2}U_{34}$ and $I_{BB'}$. Analogous operators involving the sites B and B'.

$U_{23}/q^{1/2}$: If sites A and B are coloured alike, introduce a factor 1, otherwise introduce a factor zero. Thus $R_AR_B \to R_AR_B$, $R_AY_B \to 0$, etc. (Similarly, if it acts on the pair of sites A', B it has the effect $R_{A'}R_B \to R_{A'}R_B$, $R_{A'}Y_B \to 0$, etc.) I_{AB} has the effect $R_AR_B \to R_AR_B$, $R_AY_B \to R_AY_B$, etc and acts similarly on $R_{A'}R_B$, $R_{A'}Y_B$, etc.

$U_{45}/q^{1/2}$ and I_{BC}: Similar effects on a neighbouring pair of sites B, C; B', C; etc.

Using the operators U_{12}, U_{23} and their two products, we can represent the effect of all Potts-type two-site and three-site interactions within the triangle of sites ABA', sites A and B being in the same row and site A' in the next row; Baxter *et al* (1978). For the particular case of purely three-site interaction within this triangle we found the operator

$$q^{1/2}U_{12}+gU_{23}/q^{1/2} \tag{6}$$

which gives a weight $(1+g)$ if all three sites A, A', B are coloured alike, and unity otherwise. If we multiply (6) on the left by the similar operator $q^{1/2}U_{34}+gU_{45}/q^{1/2}$, we introduce similar weightings for the triangle of sites B, B', C. Figure 4 shows that a product of such factors gives the row transfer operator for a triangular lattice with three-site interactions around all the 'up' triangles. The function ψ to be inserted into (1) is the appropriate power of the largest eigenvalue of this operator.

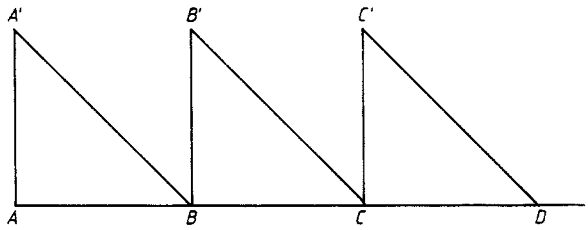

Figure 4

We have not yet solved this eigenvalue problem but we deduced two pieces of information. The U operators obey the relations

$$\begin{aligned} &U_{12}^2=q^{1/2}U_{12} && U_{23}^2=q^{1/2}U_{23}, \text{ etc} \\ &U_{12}U_{23}U_{12}=U_{12}, \text{ etc} && U_{23}U_{12}U_{23}=U_{23} \end{aligned} \tag{7}$$

(two operators without a suffix in common commute). We use the dual transformation

$$U_{12}\to U_{23}\to U_{34}\to U_{45}\ldots \tag{8}$$

which transforms (6) into the corresponding operator for the 'down' triangle $A'BB'$ if $g=q$. (As usual, we assume that this locates the transition.) For $q=1$, the critical value of g is 1 and consequently $p_c=\frac{1}{2}$ agreeing with the result obtained by Sykes and Essam (1964). The relation (123) of Baxter *et al* deduced from (8), gives us simply that at critical, the number of black sites is $\frac{1}{2}N$, which is obvious since the critical probability is $\frac{1}{2}$. Thus, it gives us no new information for the percolation case $q=1$.

We can use the properties of the U operators enumerated above to describe four-site interactions. For the square of sites $A'BB'A''$ (figure 5) we deduce the operator

$$q^{1/2}U_{12}q^{1/2}U_{34}+gU_{23}/q^{1/2} \tag{9}$$

which gives a weight $(1+g)$ if all four points are coloured alike and unity otherwise. If we now multiply this on the left by

$$q^{1/2}U_{34}q^{1/2}U_{56}+gU_{45}/q^{1/2} \tag{10}$$

we introduce a similar interaction around the square $B'CC'B''$ in figure 5. We have to start from B' rather than B, because, since U_{12} and U_{34} are both present in operator (9),

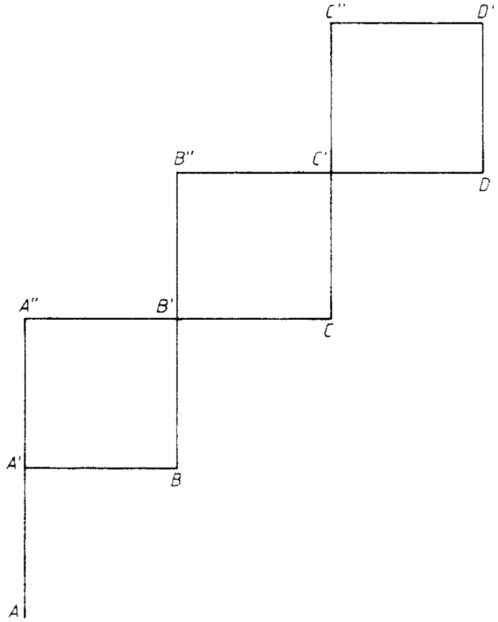

Figure 5

the effect of it has been to transfer attention from sites A, B to sites A', B', so site B' is the starting point for operator (10). That is to say we are introducing four-site interactions around alternate squares in the plane square lattice. Figure 5 shows that it is being extended towards the North-West as each row is added. Figure 1 then illustrates the correspondence between choices of black sites on L and patterns of shaded squares on L_M. Thus a product of operators such as (9) and (10) reduces to the row transfer operator related to the site problem on the plane square lattice.

The application of the dual transformation (8) to operator (6) produces an equivalent operator with, in general, a new value of g. No such equivalence applies if we apply the dual transformation (8) to operators (9) or (10). We can interpret the transformed operators. To a multiplying factor (9) transforms into

$$g' U_{23} U_{45}/q + q^{1/2} U_{34}. \tag{11}$$

Consider now a square lattice with sites labelled as in figure 6(a). Operator (11) describes interactions within the square $A'BCB'$ at the same time transferring attention from site B to site B'. As before, the operator $q^{1/2}U_{34}$ gives a weight unity whatever colour we assign to B', while the operator $U_{23}U_{45}$ gives a non-zero contribution only if sites A', B and C are all coloured alike and, when attention is transferred from B to B', such a configuration is converted into one in which A', B' and C are all coloured alike. Thus, operator (11) also introduces a weighting factor $(1+g')$ whenever the four sites A', B, C, B' are coloured alike and a factor 1 for all other colourings. If we now apply the operator $g'U_{45}U_{67}/q + q^{1/2}U_{56}$ we introduce similar weights for possible colourings of the square $B'CDC'$. That is to say that the product of operators like (11) introduces four-site interactions around *every* square, not around *alternate* squares as do products of operators like (9) and (10).

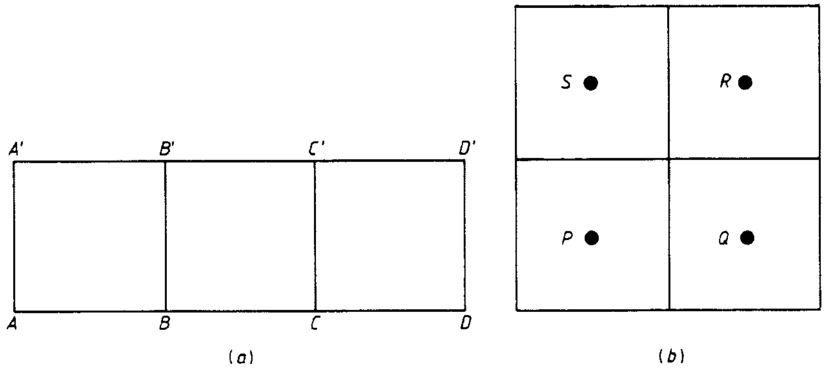

Figure 6

We can relate operator (10) to a site percolation problem by introducing a blackened site into the centre of *every* square of the plane square lattice whenever its four corners are coloured alike. Thus, in figure 6(b), if sites P and R are both blackened, the group of seven surrounding sites must be all coloured alike, so sites P and R must be connected. Similarly, the blackening of sites P and Q implies that the group of six surrounding sites must all be coloured alike, so P and Q must be joined in the percolation lattice. We conclude that the percolation problem to which a product of operators like (11) reduces as $q \to 1$ is the plane square lattice with both diagonals in every square. This is just the 'matching' lattice of the plane square lattice found by Sykes and Essam (1964). Thus their matching transformation is the particular case when $q \to 1$ of our transformation (8).

We can also write down the operator corresponding to three-site interactions around *all* triangles of the plane triangular lattice. Thus in figure 6(a), such an interaction around the triangle $A'BB'$ corresponds to the operator

$$g''U_{23}/q^{1/2}+q^{1/2}U_{34}. \tag{12}$$

The corresponding operator that produces three-site interactions around the two triangles $A'BB'$, BCB' in the square $A'BB'C$ (figure 7) is found to be

$$q^{1/2}U_{34}+gU_{23}/q^{1/2}+gU_{45}/q^{1/2}+g^2U_{23}U_{45}/q \tag{13}$$

and a product of such operators will add further squares like $B'CDC'$, etc. Interpreting this operator by inserting sites of the percolation lattice at the centre of each triangle, we arrive at the matching lattice of the plane honeycomb lattice (Sykes and Essam 1964). Applying the dual transformation (8) to the operator (13) we obtain the operator for the plane honeycomb lattice.

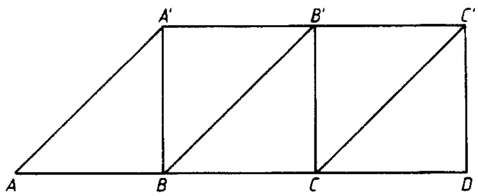

Figure 7

4. Conclusion

The operators corresponding to various site percolation problems are of relatively simple form and such problems, like bond percolation problems, can be expressed in terms of row transfer operators. Our work suggests, though it does not formally prove, that our dual transformation is the same as the Sykes and Essam matching transformation.

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